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Spatial correlations and photon counting†

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Abstract. The effect of incomplete spatial coherence on photon correlation measurements is calculated in terms of the solutions to integral equations. The N time generating function is calculated with a new approach to the 'short counting time' approximation and the two-time generating function is calculated exactly for a Lorentz spectrum. The joint counting rates for a full digital correlator are obtained and approximate solutions to the 'clipped' correlator counting rate problem are indicated.

1. Summary of photon correlation spectroscopy

In recent years it has become possible to measure the fine structure of the correlation functions of light intensities through the use of photon correlation devices. The field correlations are conventionally written as (Glauber 1963)

$$G^{(1)}(r, r', t, t') = \langle E^{(-)}(r, t)E^{(+)}(r', t')\rangle \tag{1}$$

where $E^{(-)}$ and $E^{(-)}$ are, respectively, the positive and negative parts of the electric fields. These correlation functions contain information about both the power spectrum and the physical dimensions of the source. While most recent efforts in intensity correlation spectroscopy have been directed towards the measurement of the spectral lineshapes it should be remembered that the technique was originally developed by Hanbury Brown and Twiss with the intention of measuring the dimensions of stellar sources (Hanbury Brown and Twiss 1956).

Photon correlation devices measure the correlations between photons arriving during distinct time intervals. We will see later how such a measurement of 'intensity' correlations can yield information about field correlations. A two-time correlator compares the number of counts arriving during a time interval ΔT centred on t_1 with the number of counts arriving during the time interval ΔT centred on t_2. If we define $p\Delta T \equiv t_1 - t_2$ we would measure

$$C(p) \equiv \langle N(0)N(p)\rangle \tag{2}$$

where we have defined $N(p)$ as the number of photons detected during the interval $p\Delta T < t < (p+1)\Delta T$. Alternatively, we could measure the so called 'clipped' correlation

$$C_k(p) = \frac{\langle N_k(0)N(p)\rangle}{\langle N_k(0)\rangle} \tag{3}$$

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where we have defined

$$N_k(p) = \begin{cases} 1 & \text{if } N(p) \geqslant k \\ 0 & \text{if } N(p) < k. \end{cases} \quad (4)$$

It should be noticed that the k used in this paper is one greater than the k used by Jakeman (1970).

2. Calculating the generating function for the photon counting problem in terms of the eigenvalues of an integral equation

In calculating the joint counting rates indicated in equations (2) and (3) it is most convenient to use the 'generating function' which is defined in terms of the joint probability function $P(n_1, \dots n_N)$, where we have defined $P(n_1, \dots n_N)$ as the joint probability for detecting n_1 photons during the interval ΔT centred on t_1 and detecting n_2 counts during an interval ΔT centred on t_2, and so on. The generating function is defined as follows:

$$Q(s_1, s_2, \dots s_N) = \sum_{n_1=0}^{\infty} \dots \sum_{n_N=0}^{\infty} (1-s_1)^{n_1} \dots (1-s_N)^{n_N} P(n_1, \dots n_N). \quad (5)$$

Using methods developed by Jakeman (1970) and others, we find that we can write

$$Q(s_1, \dots s_N) = \prod_{j=1} (1+\lambda_j)^{-1} \quad (6)$$

where the λ_j are the eigenvalues of the following integral equation:

$$\lambda_k \Phi_k(r, t) = S_0 \sum_{n=1}^{N} s_n \int_{t_n - \Delta T/2}^{t_n + \Delta T/2} \mathrm{d}t' \int_B \mathrm{d}^3 r' G^{(1)}(r, r', t, t') \Phi_k(r', t'). \quad (7)$$

Here S_0 is the sensitivity of the detector and B is the area of the detector.

We can write the field correlation function $G^{(1)}$ in a form convenient for the solution of equation (7) if we make a few relatively nonrestrictive assumptions: (i) we will assume that the spectrum of the scattered light incident on the detector is characterized by a spectrum $S(\omega)$ which is symmetric in frequency about a central frequency ω_0; (ii) we will suppose that the spectrum has a linewidth $\Delta\omega$ such that $s(\omega)$ is negligible outside of the range $\omega_0 - \Delta\omega < \omega < \omega_0 + \Delta\omega$; (iii) we assume that (the maximum linear dimension of the source)/$R_0 \ll 1$, where R_0 is the distance between the detector and the source; (iv) we assume that the source is stationary in space and time. When these assumptions are valid, we can write quite generally

$$G^{(1)}(r, r', t, t') = \langle |E|^2 \rangle g(\hat{y} - \hat{y}') h(r - r') \exp\{-i\omega_0(\hat{x} - \hat{x}')\} \quad (8)$$

where we have used the following definitions:

$$g(t) = \int_{-\infty}^{\infty} \frac{\exp(-i\omega t) S(\omega)\, \mathrm{d}\omega}{\langle |E|^2 \rangle}$$

$$\hat{y} = t - \frac{|r|}{c}$$

$$\langle |E|^2 \rangle = G^{(1)}(0, 0, 0, 0)$$

$$h(r-r') = \frac{G^{(1)}(r, r', 0, 0)}{G^{(1)}(0, 0, 0, 0)}$$

$$\hat{x} = t - \frac{|r|^2\omega_0}{2R_0c}. \tag{9}$$

The function $h(r-r')$ carries information about the shape of the source. If we hold r' fixed, the function $h(r-r')$ simply describes the diffraction (as measured from r') which is produced by the source. This is the theorem of van Cittert and Zirnike (see Born and Wolf 1964). If the source has an area A as seen by the detector, we can write $h(r-r')$ in the following form (Born and Wolf 1964):

$$h(r-r') = \frac{1}{A}\int_A \exp\{-\mathrm{i}\mu(r-r').R\}\,\mathrm{d}^2R \tag{10}$$

where we have defined $\mu = \omega_0/cR_0$ and have again made use of assumption (iii).

In particular, if the source is a rectangle with sides a and a'

$$h(r-r') = \frac{\sin\{\mu a(x-x')\}}{\mu a(x-x')}\,\frac{\sin\{\mu a'(y-y')\}}{\mu a'(y-y')}. \tag{11}$$

If the source is a disc with a radius a, we can again use (10) to find

$$h(r-r') = \frac{2\mathrm{J}_1\{\mu a(|r-r'|)\}}{\mu a(|r-r'|)}. \tag{12}$$

If (the maximum dimension of the detecting system)/c is small compared with the times over which $g(t)$ changes significantly, we can approximate $g(y-y')$ by $g(t-t')$. In this case the field correlation function factors into a part depending only on time and a part depending only on position. The time part contains information about the spectrum of the source while the spatial part contains information about the shape of the source. Since photon correlation experiments are most commonly concerned with either the spectrum or the source shape and not both, the spatial and temporal effects must be separated (Hanbury Brown and Twiss 1957). We will find how to do this in the next section. We will begin by finding the approximate solutions to the integral equation which hold only for short counting times and we will indicate the exact solution to the problem in the case of a Lorentz spectrum in the concluding section.

3. An approximate solution to the integral equation for the case of short counting times

If we assume that the counting time ΔT is much smaller than the time over which $g(t)$ varies significantly, we can simplify equation (7) by multiplying both sides by $\exp(-\mathrm{i}\omega_0\hat{x})$ and integrating over the interval $(t_j-\Delta T/2) < t < (t_j+\Delta T/2)$. We then find

$$\lambda_k Y_k(t_j, r) = \sum_{n=1}^{N} \bar{n}s_n \frac{1}{B}\int_B Y_k(t_n, r')h(r-r')g(t_j-t_n)\mathrm{d}^2r' \tag{13}$$

where we have defined

$$Y_k(t_n, r) = \exp\left(\frac{\mathrm{i}\omega_0|r|^2}{2cR_0}\right)\int_{t_n-\Delta T/2}^{t_n+\Delta T/2} \exp(-\mathrm{i}\omega_0 t)\Phi_k(r, t)\,\mathrm{d}t \tag{14}$$

and $\bar{n}$, the expected number of counts arriving during the interval ΔT is given by

$$\bar{n} = S_0 BT\langle|E|^2\rangle. \tag{15}$$

Since the kernel of (7) was factored by the approximation in (8) into space and time parts, we expect that the eigenfunctions can be similarly factored. We will assume that we have found the M orthonormal functions $f_k(r)$ which satisfy the equations

$$b_k f_k(r) = \frac{1}{B}\int_B h(r, r')f_k(r')\,\mathrm{d}^2r' \tag{16}$$

$$\int_B f_k^*(r)f_{k'}(r)\,\mathrm{d}^2r = \delta_{kk'} \tag{17}$$

and write

$$Y_k(t_n, r) = Y_k(t_n, 0)f_k(r). \tag{18}$$

Using (16) and (18) in (13) we then find

$$\lambda_k Y_k(t_1, 0) = \bar{n}b_k \sum_{n=1}^{N} s_n g(t_1 - t_n)Y_k(t_n, 0). \tag{19}$$

We can also write (19) as a matrix eigenvalue problem

$$\lambda_k Y_k(m) = \sum_{n=1}^{N} D_k(m, n)Y_k(n) \tag{20}$$

where

$$D_k(m, n) = \bar{n}b_k s_m g(t_m - t_n) \tag{21}$$

since the matrix D_k is of Nth order, for each eigenvalue b_k of equation (16) there will be N eigenvalues λ_k associated with equation (20). There will thus be a total of $M \times N$ solutions to the eigenvalue problem in equation (7). In most cases of interest the kernel of equation (16) will be nondegenerate and M will be infinite.

We can construct a matrix equation which gives all MN eigenvalues of (7) simultaneously if we use the $MN \times MN$ matrix which has the $N \times N$ matrices D_k along its main diagonal

$$D = \begin{pmatrix} D_1 & & & 0 \\ & D_2 & & \\ & & D_3 & \\ 0 & & & \ddots \end{pmatrix}. \tag{22}$$

The diagonalized form of D contains all of the eigenvalues λ_k. We can thus use D and equation (6) to find the generating function

$$Q(s_1 \ldots s_N) = \prod_k (1+\lambda_k)^{-1} = \{\det(1+D)\}^{-1} = \prod_k \{\det(1+D_k)\}^{-1}. \tag{23}$$

We can write the double time generating function explicitly by using equation (21)

$$Q(s_1, s_2) = \left(\prod_{j=1}^{M} \{1+\bar{n}b_j s_1 + \bar{n}b_j s_2 + (\bar{n}b_j)^2(1-|g(t_1-t_2)|^2)s_1 s_2\}\right)^{-1}. \tag{24}$$

We have used the fact that $g^*(t-t') = g(t'-t)$.

Returning to equation (16) we see that if the detector and the source are sufficiently small and sufficiently far apart so that $h(r-r') = 1$ over the entire area of the detector, there is only one nonzero eigenvalue to equation (16), that is, $b_1 = 1$. If we use this single eigenvalue in equation (23) and make use of equation (21) we find that we have recovered the short time generating function originally calculated by Bedard (1967)

$$Q(s_1 \dots s_N) = \{\det(1+D_1)\}^{-1}$$
$$D_1(m,n) = (1-\delta_{mn})s_n\bar{n}g(t_m-t_n)+\delta_{mn}(1+\bar{n}s_n).$$

4. Calculation of the joint counting rates in the short time approximation

We can now proceed to find the joint counting rates which are measured in photon correlation experiments. Making use of equations (5) and (24) we find

$$\begin{aligned}\langle N(0)N(p)\rangle &= \frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2}Q(s_1, s_2)|_{s_1=s_2=0} \\ &= \bar{n}^2\left(\left(\sum_{i=1}^{M} b_i\right)^2 + |g(p\Delta T)|^2 \sum_{i=1}^{M} b_i^2\right).\end{aligned} \tag{25}$$

The sums can be evaluated by using a well known theorem from the theory of integral equations which tells us that when $h(r-r')$ is nondegenerate we can write

$$\frac{1}{B}h(r-r') = \sum_{i=1} b_i f_i^*(r) f_i(r'). \tag{26}$$

If we now set $r = r'$ and integrate both sides of (26) over the area B we find

$$\sum_{i=1} b_i = 1. \tag{27}$$

If we multiply both sides of (26) by $h^*(r-r')$ and integrate both r and r' over the area B we find

$$\sum_{i=1} b_i^2 = \left(\frac{1}{B}\right)^2 \int_B \int_B \mathrm{d}^2r\, \mathrm{d}^2r' |h(r-r')|^2 \equiv \beta. \tag{28}$$

Using (26) and (27) in (25) we find that we have recovered the conventional form of the two-time correlation of count rates

$$\langle N(0)N(p)\rangle = n^2(1+\beta|g(p\Delta T)|^2). \tag{29}$$

Calculations of β for various forms of sources and detectors have been performed by several authors (Hanbury Brown and Twiss 1957, Scarl 1968, Jakeman *et al* 1970, Haig and Sillito 1968).

Calculations for the clipped correlator require us to obtain the eigenvalues of (16) explicitly. Slepian (1964) has studied the solutions to homogeneous integral equations of the type (16) when the kernel can be written in the form of equation (10) and when the areas of the source A and the detector B meet the following requirements: (i) A and B are scaled versions of each other, that is if $r \in A$ if and only if $(r/d) \in B$ when d is some positive constant; (ii) both A and B are symmetric, that is if $r \in A$ implies $-r \in A$. We will examine two cases which meet these requirements and which are of particular interest to light scattering problems.

When the source is a rectangle of dimensions a by a' and the detector is a rectangle of dimensions b by b', the eigenfunctions of (16) are prolate spheroidal wavefunctions of zero order. The eigenvalues can be written in Flammer's notation as follows (Flammer 1957):

$$b_{mn} = (R^{(1)}_{0n}(c_0, 1)R^{(1)}_{0m}(c'_0, 1))^2 \tag{30}$$

where we have defined $c_0 = ab\mu$ and $c'_0 = a'b'\mu$. If both the source and the detector are squares, that is, if $a = a'$ and $b = b'$, the first three eigenvalues can be approximated for small values of c_0 as follows (Slepian and Sonnenblick 1965):

$$b_1 = 1-\frac{2c_0^2}{9}+(\text{terms of order } c_0^4) \tag{31}$$

$$b_2 = b_3 = \frac{c_0^2}{9}+(\text{terms of order } c_0^4). \tag{32}$$

Notice the degeneracy of the eigenvalues due to the symmetry of the rectangle. Slepian and Sonnenblick (1965) have tabulated values of $R^{(1)}_{0n}(c_0, 1)$ for various values of c_0.

If the source is a circle with radius a and the detector is a circle with a radius a', the eigenfunctions are generalized prolate spheroidal wavefunctions and we find, following Slepian (1964), that

$$b_n = \frac{4\gamma^2}{c_0} \tag{33}$$

where $c_0 = aa'\mu$ and γ solves the following integral equation:

$$\gamma\phi(r) = \int_0^1 J_N(c_0rr')(c_0rr')^{1/2}\phi(r')\,dr'. \tag{34}$$

Slepian's paper tabulates γ^2 for various values of the argument c_0. We have converted some of Slepian's figures to notation useful to this paper and the result is shown in table 1. Two things should be noticed. First, when c_0 is small, that is when the detector spans less than a spatial 'coherence area' of the incident light, only a single eigenvalue is significant and it is, as we anticipated, close to unity. When c_0 becomes large, we find that there are a number of approximately equal eigenvalues.

Table 1. The eigenvalues for equation (16) in the case of circular and square sources and detectors.

	Discs with $c_0 = \omega_0 r_0 r'_0/cR_0$					Squares with $c_0 = \omega_0 ab/cR_0$				
c_0	1·0	2·0	3·0	4·0	5·0	1·0	2·0	3·0	4·0	5·0
b_1	0·8844	0·6296	0·3942	0·2437	0·1592	0·8090	0·4783	0·2610	0·1529	0·0985
b_2	0·0556	0·1612	0·2148	0·1962	0·1499	0·0887	0·1932	0·1899	0·1401	0·0967
b_3	0·0556	0·1612	0·2148	0·1962	0·1499	0·0887	0·1932	0·1899	0·1401	0·0967
b_4	0·0016	0·0191	0·0606	0·1007	0·1125	0·0097	0·0780	0·1382	0·1283	0·0948
b_5	0·0016	0·0191	0·0606	0·1007	0·1125	0·0017	0·0195	0·0549	0·0797	0·0789
b_6	0·0004	0·0067	0·0296	0·0669	0·0926	0·0017	0·0195	0·0549	0·0797	0·0789
b_7	0·0000	0·0002	0·0009	0·0096	0·0269	0·0002	0·0079	0·0399	0·0730	0·0744
b_8	0·0000	0·0002	0·0009	0·0096	0·0269	0·0002	0·0079	0·0399	0·0730	0·0774

We can now use these calculated eigenvalues to determine the joint counting rates for the clipped correlator. Using (5) and (24) we can write

$$\frac{\langle N_k(0)N(p)\rangle}{\langle N_k(0)\rangle} = 1+\alpha_k|g(p\Delta T)|^2 \tag{35}$$

$$\alpha_1 = \frac{\Sigma_{i=1}^{M}(\bar{n}b_i)^2/(1+\bar{n}b_i)}{\bar{n}\{\Pi_{i=1}^{M}(1+\bar{n}b_i)-1\}} \tag{36}$$

$$\alpha_2 = \frac{\Sigma_{i=1}^{M}(\bar{n}b_i)^2/(1+\bar{n}b_i)\times\Sigma_{i=1}^{M}\{\bar{n}b_i/(1+\bar{n}b_i)\}+\Sigma_{i=1}^{M}(\bar{n}b_i)^3/(1+\bar{n}b_i)^2}{\bar{n}[\Pi_{i=1}^{M}(1+\bar{n}b_i)-\Sigma_{i=1}^{M}\{\bar{n}b_i/(1+\bar{n}b_i)\}-1]} \tag{37}$$

Higher order α could be calculated similarly. We have used the first ten eigenvalues of the double disc problem to plot figure 1.

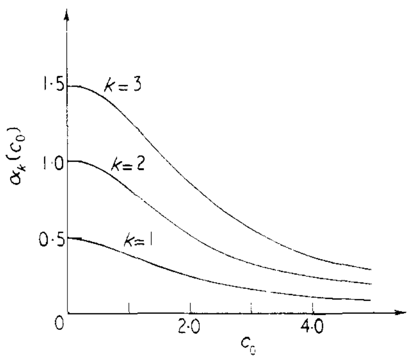

Figure 1. The values of $\alpha_k(c)$ as a function of c_0 for the case of circular sources and detectors with $\bar{n} = 1$.

From equations (29) and (35) we see that the most important effect of considering the properties of spatial coherence in a photon counting measurement designed to measure the source spectrum comes in the term which gives the relative contribution of the term $g(t)$ in the joint counting rate. By examining the calculations of β cited in the literature and by observing the curves shown in figure 1 we see that the contribution of $g(t)$ relative to the constant background diminishes as c_0 increases (it should be noticed that the number $c_0^2/4$ is conventionally called the number of 'coherence areas' spanned by the detector).

5. The exact solution in the case of a Lorentz spectrum

If the light incident on the detector has a lorentzian lineshape, the time part of the integral equation (7) can be solved exactly (Jakeman 1970). The trick is to reduce the problem to the solution of a simple differential equation whose solutions are exponentials. When we are calculating the two-time generating function, the boundary conditions on the differential equations permit solutions only when

$$\prod_{j=1}^{M}(A_k(s_1)A_k(s_2)-B_k(s_1)B_k(s_2)) \equiv F(z) = 0 \tag{38}$$

where we define

$$A_k(s_1, z) = e^{-\tau}(\cosh y_k(s, z)) + \frac{1}{2}\left(\frac{\tau}{y_k(s, z)} + \frac{y_k(s, z)}{\tau}\right) \sinh(y_k(s, z))$$

$$B_k(s, z) = \frac{1}{2} g(t_1 - t_2)\left(\frac{\tau}{y_k(s, z)} - \frac{y_k(s, z)}{\tau}\right) \sinh(y_k(s, z)) \qquad (39)$$

$$\tau \equiv \Gamma T$$

$$y_k(s, z) = \tau^2 + 2\tau s b_k \langle |E|^2 \rangle z.$$

Γ is the halfwidth of the Lorentz spectrum.

If we expand $F(z)$ as a product of its zeros (which are the eigenvalues of equation (7)), we find that we can write

$$Q(s_1, s_2) = (F(1))^{-1}. \qquad (40)$$

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References

Bedard G 1967 *Phys. Rev.* **161** 1304–8.
Born M and Wolf E 1964 *Principles of Optics* (New York: Macmillan) pp 508–20.
Flammer C 1957 *Spheroidal Wavefunctions* (Stanford: Stanford University Press) pp 6–15.
Glauber R J 1963 *Phys. Rev.* **130** 2529–39.
Haig N D and Sillitto R M 1968 *Phys. Lett.* **28A** 463–4.
Hanbury Brown R and Twiss R Q 1956 *Nature* **178** 1046–8.
—— 1957 *Proc. R. Soc.* A **243** 291–319.
Jakeman E 1970 *J. Phys.* A: *Gen. Phys.* **3** 201–15.
Jakeman E, Oliver C J and Pike E R 1970 *J. Phys.* A: *Gen. Phys.* **3** L45.
Scarl D 1968 *Phys. Rev.* **175** 1661–8.
Slepian D 1964 *Bell Syst. tech. J.* **6** 3009–56.
Slepian D and Sonnenblick E 1965 *Bell Syst. tech. J.* **7** 1745–59.